# Some useful procedures towards consistent preference modeling *

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## Abstract

Decision making based upon valued preference relations is assuming that each decision maker is able to consistently manage intensity values for preferences, but this is indeed a difficult task, even when dealing with few alternatives. Representation tools will therefore play a key role in order to help decision makers to understand their preference structure. This paper introduces a particular representation based upon classical crisp dimension theory, addressing some associated computational complexity problems, which will hopefully be useful within a valued framework.

Keywords: consistent preference representation, dimension theory.

## 1 Introduction.

Valued preferences are indeed a promising tool for modeling. In fact, classical crisp preferences force decision makers to choose the best alternative among each pair of alternatives, giving the same output no matter if such a preference is strongly supported or weakly supported. Moreover, classical decision procedures use to impose decision makers some *consistency* restrictions in their preferences, transitivity among them. But managing transitive crisp preferences is a difficult task, so quite often decision maker preferences require some numerical manipulation, perhaps in order to get an alternative preference relation being transitive and *not too far away* from the original data provided by the decision maker. These difficulties become extreme when dealing with valued preferences, in such a way that it is unrealistic to assume almost any kind of *a priori* consistency. A first approach could be a search for a consistent valued preference being *close enough* to the original preference intensities, as provided by decision makers (see, e.g., [4]). An alternative approach is to trie an informative representation, which will hopefully help decision maker to manage the data set, showing main characteristics of preferences, including possible inconsistencies. Best results are expected to be obtained by means of some mixture of both approaches, leading to a *good enough representation* of a *good enough data approximation.*

In this paper we explore a heuristic approach to some key consistency problems, to be taken into account in the particular representation model already proposed by the authors in [2, 3], where a general representation of crisp binary relations leads to a *generalized dimension theory* related to classical *dimension theory*, initially introduced by Dushnik and Miller [1] in the context of crisp partial order sets (see also Trotter [7]). Following [2, 3], for any strict valued preference relation we can analyze the sequence of its $\alpha$-cuts

$$R^\alpha = \{(x_i, x_j)/\mu(x_i, x_j) \geq \alpha\}, \quad \alpha \in (0, 1]$$

in terms of an associated general representation for

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arbitrary preference relations, not restricted to partial order sets. But since the approach initially proposed by the authors in [2, 3] implies taught computational problems, in this paper we shall consider an alternative representation which will allow approximate representations, easier to be found and evaluated.

## 2 The model.

Given a finite set of alternatives

$$X = \{x_1, x_2, \ldots, x_n\}$$

a strict crisp binary preference relation $R \subset X \times X$ can be characterized by a mapping

$$\mu^R : X \times X \to \{0, 1\}$$

such that $\mu^R(x_i, x_i) = 0$ for all $x_i \in X$, being $\mu^R(x_i, x_j) = 1$ whenever $x_i R x_j$ and $\mu^R(x_i, x_j) = 0$ otherwise. Hence, a $n \times n$ matrix $\mu$ is being defined, being $\mu_{i,j} = \mu^R(x_i, x_j), \forall i, j$. A standard *consistency* assumption is to impose that our crisp preference relation is a partial order set (poset), i.e., it is non reflexive ($\mu^R(x_i, x_i) = 0, \forall x_i \in X$), antisymmetric ($\mu^R(x_i, x_j) = 1 \Rightarrow \mu^R(x_j, x_i) = 0$), and transitive ($\mu^R(x_i, x_j) = \mu^R(x_j, x_k) = 1 \Rightarrow \mu^R(x_i, x_k) = 1$). Under these three conditions, our crisp preference relation can be represented as the intersection of linear orders, and the minimal number of these linear orders is the classical dimension of a partial order set (Dushnik and Miller [1]). But checking transitivity may not become an easy task. In fact, the deep decision making problems that may appear when the basic information is given in terms of a crisp preference relation are mostly associated to the existence of cycles (a cycle is a subset of alternatives $\{x^1, x^2, \ldots, x^k\} \subset X$ such that $x^1 R x^2 R \cdots R x^k R x^1$; see, e.g., [5, 6]). In this sense, non reflexivity avoids cycles with only one alternative, antisymmetry avoids cycles with two different alternatives, and transitivity avoids cycles involving three different alternatives. But although consistency would require the analysis of all possible cycles, an exhaustive analysis of non-transitive triples may offer a quite complete view of consistency problems.

The above arguments lead the authors to search for an alternative representation of strict preference relations, in terms of union of intersections of linear orders ([2, 3]), which may imply serious computational difficulties. Hence, we shall be considering here an alternative representation, still valid to any arbitrary crisp binary relation being non reflexive, as an approximation to the one given by the authors in [2, 3].

In particular, we know that any arbitrary strict crisp binary relation $R$ can be decomposed as union and intersections of linear orders (see [2, 3])

$$R = \bigcup_{s=1}^{h} \bigcap_{t=1}^{d_s} L_{st}$$

but we do not need to search for the minimum number of linear orders $L_{st}$ allowing such a representation (as proposed in [2, 3]), but for perhaps a more natural representation: trie to minimize the number of unions. As pointed out in [2], we may need less linear orders representing a partial order set by splitting it into the union of two partial order sets.

In other words, we now propose to represent each arbitrary non reflexive crisp preference relation as

$$R = \bigcup_{s=1}^{h} P_s$$

where parameter $h$ is being made as small as posible, being

$$P_s = \bigcap_{t=1}^{d_s} L_{st}$$

a poset of dimension $d_s$, resulting from the intersection operator on $d_s$ linear orders. In general, parameter $h$ will give the minimum number of partial order sets allowing the above representation (in case $R$ is a partial order set, then $h = 1$). Of course the number of linear orders being involved may not be minimal in the sense of [2, 3], but it gives a bound, easier to evaluate if the algorithm developed by Yáñez-Montero [9] is taken into account (see also [8]).

Anyway, in order to compute parameter $h$ we propose a search for non-transitive triples in our preference relation: let

$$INT(R) = \{(x_i, x_j, x_k) \in X \times X \times X \ / \\ x_i > x_j \ x_j > x_k \ x_i \not> x_k\}$$

be the non-transitive triples in $R$, and let $nint = |INT(R)|$ be its cardinal.

# 3 Computing procedures.

Of course, $INT(R) = \emptyset$ whenever $R$ is a transitive binary relation. A natural algorithm with computational complexity $O(n^3)$) will compute the set $INT(R)$.

Then the following properties and definitions will be the basis for a procedure to compute parameter $h$.

**Theorem 1** *Let $R$ be a binary relation without cycles such that $INT(R) = \emptyset$, then $R$ is a poset and $h = 1$.*

**Theorem 2** *Let $R$ be a binary relation with some cycles and verifying $INT(R) = \emptyset$, then $h = 2$.*

**Definition 1** *Let us denote*

$$INT(R) = \{(z^1(t), z^2(t), z^3(t)) \in X \times X \times X \ / \\ t \in \{1, 2, \ldots, nint\}\}$$

*Then we define the intransitive pairs set as the set*

$$V_I(R) = \{v(t) \ / \ t \in \{1, 2, \ldots, nint\}\} \\ \bigcup \{w(t) \ / \ t \in \{1, 2, \ldots, nint\}\}$$

*where*

$$v(t) \equiv (z^1(t), z^2(t)) \\ w(t) \equiv (z^2(t), z^3(t))$$

*for all $t \in \{1, 2, \ldots, nint\}$.*

**Definition 2** *Given a non-transitive binary relation $R$ defined on $X$, its* intransitive pairs graph *is the non-directed graph*

$$G_I(R) = (V_I(R), E_I(R))$$

*where*

$$E_I(R) = \{\{v(t), w(t)\} \ /t \in \{1, 2, \ldots, nint\}\} \\ \bigcup \{\{v(t), -v(t)\} \ / \ \textit{ if } v(t), -v(t) \in V_I(R)\}$$

Hence, parameter $h$ of a non-transitive binary relation can be computed by means of the following procedure:

1. Let $C : V_I(R) \longrightarrow \{1, 2, \ldots, h\}$ be a coloring function associated to the chromatic number of the intransitive pairs graph, $h = \chi(G_I(R))$.
2. Then, $h$ binary relations
$$\{(X, P_1), \ldots, (X, P_h)\}$$
can be defined in the following way:
   (a) For any $(x_i, x_j) \in V_I(R)$, if the color of this node is $C((x_i, x_j)) = c$, with $c \in \{1, \ldots, h\}$, then the relation $x_i > x_j$ is introduced in the relation $P_c$.
   (b) The binary relations $P_c$ so defined does not contain any opposite pairs, by the definition of $E_I(R)$.
   (c) To assure transitivity, these relations are *transitivized*, in the sense that relations in $P_c$ are substituted by a transitive binary relation $tr(P_c)$ (see theorem 3 below).
3. In order to put the relation $R$ as the union of $h$ posets, those pairs included in $R$ but not included in any of the relations $tr(P_c)$ can be taken into account: let
$$R^{ct} = R - \bigcup_{c=1}^{h} tr(P_c)$$
In this way, all pairs of $R^{ct}$ can be included in some relation $tr(P_c)$, keeping transitivity (see theorem 4 and theorem 5). If we let $P'_c$ be the poset defined by the extended relation $tr(P_c)$ with those pairs of $R^{ct}$, then the non-transitive binary relation $R$ can be decomposed as union of posets
$$R = \bigcup_{c=1}^{h} P'_c$$

**Theorem 3** *Let $R$ be a non-transitive binary relation defined on $X = \{1, \ldots, n\}$ and let $C$ be a coloring function attaining the chromatic number*

$$h = \chi(G_I(R))$$

*Then, the binary relations $P_c$ defined as*

$$P_c = \{(i, j) \in V_I(R)/C((i, j)) = c\}, \forall c \in \{1, \ldots, h\}$$

*verify*

$$tr(P_c) - P_c \subset R \quad \forall c \in \{1, \ldots, h\}$$

*where $tr(P_c)$ is a minimal* transitivized *relation induced by $P_c$.*

**Theorem 4** *Any intransitive chain belonging to $R^{ct}$ can be removed from this set.*

**Theorem 5** *The pairs of $R^{ct}$ can be moved to some $P_c$.*

**Theorem 6** *Given an arbitrary intransitive binary relation $R$, i.e. $INT(R) \neq \emptyset$, the parameter $h$ of $R$ verifies $h = \chi(G_I(R))$.*

# 4 Final comments.

Summarizing results, due to theorem 1, theorem 2 and theorem 6, the following scheme can be considered in order to compute the parameter $h$ of an arbitrary relation $R$:

- If $nint = 0$, then
  - either $\nexists$ cycles in $R \Rightarrow \; h = 1$;
  - or $\exists$ cycles in $R \Rightarrow \; h = 2$.
- Otherwise, $nint > 0 \Rightarrow h = \chi(G_I(R))$.

Therefore, by means of this parameter $h$ we shall be able to decompose any crisp strict preference relation into the union of several partial order sets, and the classical dimension of each one of them can be tried by applying the algorithm of Yáñez-Montero [9]. In this way, if we consider arbitrary valued preference relations

$$\mu : X \times X \to [0, 1]$$

with the only restriction $\mu(x_i, x_i) = 0$ for all $x_i \in X$, we shall be able to get a representation in terms of the union of intersection of linear order, for every $\alpha$-cut $R^\alpha, \alpha \in (0, 1]$, at least for medium size problems. It may be not an optimal representation in the sense of [2, 3], but it may be considered an operational approximation.

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